<table>
<thead>
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<th>TIME LINE</th>
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<td><strong>Counting Numbers</strong></td>
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<td>Prehistoric</td>
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<td>800 Hindu/Arabic</td>
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Isaac Newton
1643-1727
Every year about a million American students take calculus courses.

But few really understand what the subject is about or could tell you why they were learning it.

It’s not their fault.

There are so many techniques to master and so many new ideas to absorb that the overall framework is easy to miss.

The JOY of Calculus
Double, double, toil and trouble;
Fire burn and cauldron bubble.
Eye of newt, and toe of frog,
Wool of bat and tongue of dog;
Adder's fork and witches shawls,
Chicken soup and matzo balls.

Double, double, toil and trouble;
Limits are the limits.
Question: Does the number $0.999\ldots = 1$?
LIMITS

Consider the sequence \( x_1 = .9 \)
\( x_2 = .99 \)
\( x_3 = .999 \)
\( \ldots \)
\( x_n = \underbrace{.999\ldots 9}_{n \text{ 9's}} \)

As \( n \) gets larger and larger, \( x_n \) gets closer and closer to 1.
In mathematical symbols, we write

$$\lim_{n \to \infty} x_n = 1$$

which is read, “the limit of \( x_n \) as \( n \) goes to infinity is 1.”

Three things to notice:

1) \( x_n < x_{n+1} \) foreach \( n \)

2) no \( x_n \) is actually equal to the number 1. It is the *limit* that is 1.

3) Every circle with center at 1, *no matter how small*, contains infinitely many of the \( x_n \)’s.
Question: Does the number $0.999\ldots = 1$?
The idea that as a sequence of numbers get closer and closer to one particular number was used by the great Archimedes thousands of years ago to compute the value of \(\pi\).
Me and my pal Archimedes
Archimedes approximated a circle by a polygon with many straight sides, and then kept doubling the number of sides to get closer to perfect roundness.
Then he used the Pythagorean theorem to work out the areas of these inner and outer polygons, starting with the hexagon and bootstrapping his way up to 12, 24, 48 and ultimately 96 sides. The results for the 96-gons enabled him to arrive at the well-known formula that the area of a circle is

$$A = \pi r^2$$
Johannes Kepler
1571 - 1630
Figure 1: Illustration of Kepler's three laws with two planetary orbits. (1) The orbits are ellipses, with focal points $f_1$ and $f_2$ for the first planet and $f_1$ and $f_3$ for the second planet. The Sun is placed in focal point $f_1$. (2) The two shaded sectors $A_1$ and $A_2$ have the same surface area and the time for planet 1 to cover segment $A_1$ is equal to the time to cover segment $A_2$. (3) The total orbit times for planet 1 and planet 2 have a ratio $a_1^{3/2} : a_2^{3/2}$. 
$y = 2x + 1$

Area is 10

(Work on Board)
What’s so exciting about this calculation is the way infinity and limits come to the rescue. At every finite stage, the rectangular shapes are rough approximations to the area. But when you take it to the limit as the number of rectangles goes to infinity, it becomes exact and beautiful and everything becomes clear. That’s how calculus works.
BUT THAT’S A LOT OF WORK!!!
\[ \lim_{x \to 2} (3x + 7) = 13 \]

\[ \lim_{x \to 3} \frac{x - 9}{x - 3} \text{ does not exist} \]

\[ \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6 \]
The Speedometer Problem

- Average speed over a given time period
- Speed at a given instant
Definition:

Speed is the *rate of change* of position (or think of it as *rate of change* of distance if you prefer) with respect to time.
Suppose an object is moving along a path and that \( s(t) \) is its position at time \( t \).

Now let \( t_0 \) be a particular moment and we want to find the speed of the object at time \( t_0 \).

For any number \( h \neq 0 \), the average speed over the time interval from \( t_0 \) to \( t_0 + h \) is

\[
\frac{[s(t_0 + h) - s(t_0)]}{h}
\]

and we define the speed at time \( t_0 \) to be the limit of these average speeds over smaller and smaller time intervals; i.e,
That is, the speed of the object at time $t_0$ is

$$\lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

The speed is rate of change of position with respect to time.
Example: Suppose an object is moving along a line and that its position $s$ at any time $t$ is given by $s(t) = 2t$. Find its speed at time $t = 3$.

Solution: According to our definition, the speed of the object at $t = 3$ is

$$\lim_{h \to 0} \frac{s(3 + h) - s(3)}{h}$$

$$= \lim_{h \to 0} \frac{2(3 + h) - 2 \cdot 3}{h}$$

$$= \lim_{h \to 0} \frac{6 + 2h - 6}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h}$$

$$= 2$$