OK: Now we are ready to INTEGRATE !!!

OR ARE WE?

(Board)
Here’s a fun game: Given a function $f$, guess a function $F$ such that $F’ = f$ and then check your answer to make sure its right.

Example: Let $f(x) = 5$. Can you guess a function $F$ whose derivative is 5?

Of course you can; try $F(x) = 5x$. Check: $F’(x) = 5$.

Question: Is $F(x) = 5x$ the only function whose derivative is 5? What about $G(x) = 5x + 7$? Or $H(x) = 5x - \pi$?

Harder Example: Let $f(x) = 2x - 3$. Can you guess a function whose derivative is $2x - 3$?

How about $F(x) = x^2 - 3x$? Check: $F’(x) = 2x - 3$; OK, but once again, if $C$ is ANY constant, then $G(x) = x^2 - 3x + C$ also works.
DEFINITION: Let $f$ be a function. A function $F$ is an **antiderivative** of $f$ if $F^\prime = f$. If $F$ is an antiderivative of $f$, we say that $F + C$, where $C$ represents any constant, is the **general antiderivative** of $f$.

Clearly, any two functions that differ by a constant have equal derivatives. But the converse is also true!

**EQUAL DERIVATIVES THEOREM:** If $F^\prime(x) = G^\prime(x)$, then $G(x) = F(x) + C$ where $C$ is some constant.

**NOTE:** I have hedged in the statement above; the real one involves open and closed intervals and continuity.

More Examples: Let $f(x) = 3x$. What now?

Right!! Let $F(x) = (3/2)x^2 + C$. Then $F^\prime(x) = 2(3/2)x = 3x$. 
The last example can be generalized: If \( f(x) = ax^n \) and \( n \neq -1 \), then \( F(x) = \frac{a}{n+1}x^{n+1} + C \) is the general antiderivative of \( f \). If \( n = -1 \), then \( F(x) = a(\ln x) + C \).

1) If \( f(x) = 7x^3 - 4x^2 + x - 8 \), then

\[
F(x) = \left(\frac{7}{4}\right)x^4 - \left(\frac{4}{3}\right)x^3 + \left(\frac{1}{2}\right)x^2 - 8x + C
\]

is the general antiderivative of \( f \). CHECK!!

2) If \( f(x) = \sqrt{x} + 6/x \), first rewrite it with exponents:

\[
= x^{1/2} + 6x^{-1}
\]

Now use the above technique:

\[
F(x) = \frac{2}{3}x^{3/2} + 6 \ln x + C
\]

and CHECK!!

3) What if \( f(x) = e^{x/3} \)? Well, the exponential is its own derivative so let’s try \( F(x) = e^{x/3} \) and check.

Correct answer: \( F(x) = 3e^{x/3} + C \) CHECK!!
Antiderivatives are closely related to Integration. In fact, if \( f \) is a function, we often write

\[
\int f(x) \, dx
\]

which is read “the (indefinite) integral of \( f \) of \( x \) \( dx \)” to denote the general antiderivative of \( f \). So if \( F \) is any antiderivative of \( f \), we can write

\[
\int f(x) \, dx = F(x) + C
\]

For example,

\[
\int (2x + 4) \, dx = x^2 + 4x + C \quad \text{(CHECK!!)} \quad \text{and}
\]

\[
\int (x - e^{3x} - \frac{1}{x}) \, dx = \frac{1}{2}x^2 - \frac{1}{3}e^{3x} - \ln x + C
\]
A Short Table of Integrals

\[ \int a \ f(x) \ dx = a \int f(x) \ dx \]

\[ \int [f(x) + g(x)] \ dx = \int f(x) \ dx + \int g(x) \ dx \]

If \( n \neq -1 \), then \( \int ax^n \ dx = \left[ \frac{a}{n + 1} \right]x^{n + 1} + C \)

If \( n = -1 \), then \( \int (a/x) \ dx = a \ln x + C \)

\[ \int e^{ax} \ dx = \left( \frac{1}{a} \right)e^{ax} + C \]
Any equation that involves the derivative of an unknown function is called a **differential equation**. Any function that satisfies the equation is a **solution**. To **solve** a differential equation means to find *all* solutions and the set of all solutions is called the **general solution**.

Example. Solve the equation \( y' = \frac{1}{x^2} + \sqrt{x} \). This is of the form \( y' = f(x) \). The solution, of course, is the general antiderivative of \( f \); in this case,

\[
f(x) = \frac{1}{x^2} + \sqrt{x} = x^{-2} + x^{1/2}.
\]

Therefore, \( y = -x^{-1} + \left(\frac{2}{3}\right)x^{3/2} + C \) is the answer. CHECK!
An Application to Motion
Problem: A projectile is fired straight up with an initial velocity of 256 ft/sec. How high will it rise and when will it strike the ground?

Solution: Let $s$ be its position function; that is, $s(t)$ is its position at time $t$. Then $v = s'(t)$ is its velocity function and $a = v'(t)$ is its acceleration function. We take $s(0) = 0$ and the positive direction of motion to be up. Thus gravity works against the motion (slowing down the projectile) so the acceleration due to gravity is the Newton Constant -32. That is, $a = v'(t) = -32$ so $v = -32t + C$. We are given that $v(0) = 256$ so $256 = -32(0) + C$ and it follows that $C = 256$. Now we know that $v = s'(t) = -32t + 256$ so $s(t) = -16t^2 + 256t + C$. Since $s(0) = 0$, we see that $C = 0$ and $s(t) = -16t^2 + 256t$. 
So we have all the information we need:

\[ a(t) = -32 \]
\[ v(t) = -32t + 256 \quad \text{and} \]
\[ s(t) = -16t^2 + 256t \]

1) How high will it rise? At the highest point, the velocity is zero. So set \(0 = v(t) = -32t + 256\) and solve: \(t = 8\). It takes 8 seconds to reach its maximum height which is

\[
s(8) = -16(8)^2 + 256(8)
= -1024 + 2048 = 1024 \text{ ft}
\]

2) When will it strike the ground? We can reason two ways: a) It takes 8 seconds to go up, so it takes 8 seconds to come down. Ans: In 16 seconds. Or b) When it strikes the ground, \(s(t) = 0\); so set \(s = 0\) and solve:

\[
0 = s(t) = -16t^2 + 256t = t(-16t + 256) \quad \text{so} \quad t = 0 \text{ or } 16
\]
Some Problems

1) A bullet is shot straight up with a muzzle velocity of 320 ft/sec. How high will it rise and when will it strike the ground?

2) Jim is driving his car at 60mph and suddenly applies the brakes which cause a deceleration of 7200 mi/hr/hr (i.e., an acceleration of -7200 mi/hr/hr.) GUESS how long it will take to stop and how far he will travel. Then check your estimate.
3) A landing jet has a velocity of 300mph = (440ft/sec). If a deceleration of 40 ft/sec/sec is applied immediately upon landing, what is the minimum length of runway needed to bring the plane down to the manageable speed of 40ft/sec?

4) A calculus student drops a small stone from the roof of a building and hears the sound of impact 3 seconds later. She then calculates the height of the building. Can you? (Note: Sound travels about 1,100 ft/sec in air)

5) A car is traveling 50 mph in a 25 mph zone. An officer in a patrol car starts from rest as the speeder passes him and accelerates at the rate of 5 mi/hr/sec. How long does it take the officer to catch the speeder?

Answers: 1) 1600 ft.; 20 sec.  2) 30 sec.; 1/4 mile.  3) 2400ft   4) 133 ft.    5) 20 sec.
Harder Example: Solve \( y' = 3y \).

Here the rate of change of the unknown function \( y \) is proportional to the itself. This is the case in the study of population growth and radioactive decay.

Divide both sides by \( y \) to obtain \( y'/y = 3 \). Does that help? The left side is the derivative of \( \ln y \) (chain rule) and the right side is the derivative of \( 3x \) so \( \ln y = 3x + C \) and that means \( y = e^{3x + C} = e^C e^{3x} \). Check: \( y' = 3e^C e^{3x} = 3y \).
An Application to Population Growth

Problem: The world population increased from 4.5 billion in 1980 to around 6.1 billion in 2000. Assuming that conditions remain the same, what will the population be in 2025?

Solution: We let \( y(t) \) = population at time \( t \) starting in 1980; i.e., \( y(0) = 4.5 \) (billion). We know that the rate of change of population is proportional to its current size so \( y' = ky \) and as we have just seen, the solution to this differential equation is

\[
y = y(0)e^{kt} = 4.5e^{kt}
\]

We can now use the fact that when \( t = 20 \) (the year 2000), the population is 6.1 to find the number \( k \) and then we can find \( y(45) \) which is the population in 2025.
We know $y = 4.5e^{kt}$ and $y(20) = 6.1$ so

$$6.1 = 4.5e^{20k} \quad \text{or} \quad 6.1/4.5 = e^{20k}$$

and taking the natural log of both sides, we have

$$\ln(6.1/4.5) = 20k$$

and it follows that

$$k = \frac{1}{20} \ln(6.1/4.5) \approx 0.0152$$

Therefore,

$$y \approx 4.5e^{0.0152t} \quad \text{and}$$

$$y(45) \approx 4.5e^{(0.0152)(45)} \approx 8.9 \text{ billion}$$
An Application to Carbon 14 Dating

Problem: Radioactive Carbon 14 has a half-life of 5,740 years. If a fallen tree from the eruption that formed Crater Lake has only 44% of its original Carbon 14, how old is Crater Lake?

Solution: We let $y(t) = \text{the amount of carbon 14 in the tree at time } t$ with $y(0) = \text{original amount}$. We know that the rate of change in the amount is proportional to the amount present so $y' = ky$ and as before

$$y = y(0)e^{kt}$$

We are not given $y(0)$ but since the half-life is 5,740 we can write

$$0.5y(0) = y(0)e^{5740k}$$

and taking the natural log of both sides, we have

$$\ln 0.5 = 5740 k$$

so

$$k = \frac{\ln 0.5}{5740}$$
So now our equation looks like

\[ y = y(0)e^{\left(\frac{\ln 0.5}{5740}\right) t} \]

and we want to know the value of \( t \) when \( y = 0.44 \, y(0) \).

So set \( 0.44 \, y(0) = y(0)e^{\left(\frac{\ln 0.5}{5740}\right) t} \) and take the log of both sides to obtain

\[ \ln 0.44 = \left(\frac{\ln 0.5}{5740}\right) t \]

Now solve for \( t \)

\[ t = \frac{(5740)(\ln 0.44)}{(\ln 0.5)} \approx 6,799 \text{ years} \]