OK: Now we are ready to INTEGRATE !!!

OR ARE WE ?

(Board)
Any equation that involves the derivative of an unknown function is called a **differential equation**. Any function that satisfies the equation is a **solution**. To **solve** a differential equation means to find *all* solutions and the set of all solutions is called the **general solution**.
Example. Solve the equation \( y' = 1/x^2 + \sqrt{x} \). This is of the form \( y' = f(x) \). The solution, of course, is the general antiderivative of \( f \); in this case,

\[
f(x) = 1/x^2 + \sqrt{x} = x^{-2} + x^{1/2}.
\]

Therefore, \( y = -x^{-1} + (2/3)x^{3/2} + C \) is the answer. CHECK!
An Application to Motion

Problem: A projectile is fired straight up with an initial velocity of 256 ft/sec. How high will it rise and when will it strike the ground?

Solution: Let \( s \) be its position function; that is, \( s(t) \) is its position at time \( t \). Then

\[ v = s'(t) \text{ is its velocity function and} \]

\[ a = v'(t) \text{ is its acceleration function.} \]

We take \( s(0) = 0 \) and the positive direction of motion to be up. Thus gravity works against the motion (slowing down the projectile) so the acceleration due to gravity is the Newton Constant -32.
That is, 

\[ a = v'(t) = -32 \quad \text{so} \quad v(t) = -32t + C \]

We are given that \( v(0) = 256 \) so

\[ 256 = -32 \cdot 0 + C \]

and it follows that \( C = 256 \).

Now we know that

\[ v = s'(t) = -32t + 256 \quad \text{so} \quad s(t) = -16t^2 + 256t + C \]

Since \( s(0) = 0 \), we see that \( C = 0 \) and \( s(t) = -16t^2 + 256t \).
So we have all the information we need:

\[ a(t) = -32 \]
\[ v(t) = -32t + 256 \quad \text{and} \]
\[ s(t) = -16t^2 + 256t \]

1) How high will it rise?

At the highest point, the velocity is zero. So set

\[ 0 = v(t) = -32t + 256 \]
and solve: \( t = 8 \).

It takes 8 seconds to reach its maximum height which is

\[ s(8) = -16(8)^2 + 256(8) \]

\[ = -1024 + 2048 = 1024 \text{ ft} \]
2) When will it strike the ground?

We can reason two ways:

a) It takes 8 seconds to go up, so it takes 8 seconds to come down. Ans: In 16 seconds. Or

b) When it strikes the ground, \( s(t) = 0 \); so set \( s = 0 \) and solve:

\[
0 = s(t) = -16t^2 + 256t
\]

\[
= t(-16t + 256) \text{ so } t = 0 \text{ or } 16
\]
Some Problems

1) A bullet is shot straight up with a muzzle velocity of 320 ft/sec. How high will it rise and when will it strike the ground?

Ans. 1600 ft.; 20 sec.

2) Jim is driving his car at 60mph and suddenly applies the brakes which cause a deceleration of 7200 mi/hr/hr (i.e., an acceleration of -7200 mi/hr/hr.) GUESS how long it will take to stop and how far he will travel. Then check your estimate. Ans. 30 sec; 1/4 mile
3) A landing jet has a velocity of 300mph = (440ft/sec). If a deceleration of 40 ft/sec/sec is applied immediately upon landing, what is the minimum length of runway needed to bring the plane down to the manageable speed of 40ft/sec? 
Ans. 2400 ft.

4) A calculus student drops a small stone from the roof of a building and hears the sound of impact 3 seconds later. She then calculates the height of the building. Can you? (Note: Sound travels about 1,100 ft/sec in air) 
Ans. 133 ft.

5) A car is traveling 50 mph in a 25 mph zone. An officer in a patrol car starts from rest as the speeder passes him and accelerates at the rate of 5 mi/hr/sec. How long does it take the officer to catch the speeder? 
Ans. 20 sec.
Harder Differential Equation: Solve $y' = 3y$.

Here the rate of change of the unknown function $y$ is proportional to the itself. This is the case in the study of population growth and radioactive decay.

Divide both sides by $y$ to obtain $y'/y = 3$. Does that help? The left side is the derivative of $\ln y$ (chain rule) and the right side is the derivative of $3x$ so $\ln y = 3x + C$ and that means $y = e^{3x + C} = e^C e^{3x}$. Check: $y' = 3e^C e^{3x} = 3y$.

But what is this $e^C$? It is just a constant. In some problems, we know the initial value of $y$; that is, $y(0)$. In this case we can determine the constant.

$$y(0) = e^C e^{3\cdot 0} = e^C$$

and then we have

$$y = y(0)e^{3x}$$

Very Important
An Application to Population Growth

Problem: The world population increased from 4.5 billion in 1980 to around 6.1 billion in 2000. Assuming that conditions remain the same, what will the population be in 2025?

Solution: We Let $y(t) =$ population at time $t$ starting in 1980; i.e., $y(0) = 4.5$ and $y(20) = 6.1$ (billion). We want to find $y(45)$. We know that the rate of change of population is proportional to its current size so $y' = ky$ and as we have just seen, the solution to this differential equation is

$$y = y(0)e^{kt} = 4.5e^{kt}$$

We can now use the fact that when $t = 20$ (the year 2000), the population is 6.1 to find the number $k$ and then we can find $y(45)$ which is the population in 2025.
We know \( y = 4.5e^{kt} \) and \( y(20) = 6.1 \) so

\[
6.1 = 4.5e^{20k} \quad \text{or} \quad \frac{6.1}{4.5} = e^{20k}
\]

and taking the natural log of both sides, we have

\[
\ln\left(\frac{6.1}{4.5}\right) = 20k \quad \text{and it follows that}
\]

\[
k = \left(\frac{1}{20}\right)\ln\left(\frac{6.1}{4.5}\right) \approx 0.0152
\]

Therefore,

\[
y \approx 4.5e^{0.0152t} \quad \text{and}
\]

\[
y(45) \approx 4.5e^{(0.0152)(45)} \approx 8.9 \text{ billion}
\]
Problem: Radioactive Carbon 14 has a half-life of 5,740 years. If a fallen tree from the eruption that formed Crater Lake has only 44% of its original Carbon 14, how old is Crater Lake?
Solution: We let $y(t) = \text{the amount of carbon 14 in the tree at time } t$ with $y(0) = \text{original amount}$. We know that the rate of change in the amount is proportional to the amount present so $y' = ky$ and as before

$$y = y(0)e^{kt}$$

We are not given $y(0)$ but since the half-life is 5,740 we can write

$$0.5y(0) = y(0)e^{5740k}$$

and taking the natural log of both sides, we have

$$\ln 0.5 = 5740k \quad \text{so} \quad k = \frac{(\ln 0.5)}{5740}$$
So now our equation looks like

\[ y = y(0)e^{\left(\ln 0.5\right)/5740 \ t} \]

and we want to know the value of \( t \) when \( y = 0.44 \ y(0) \).

So set

\[ 0.44 \ y(0) = y(0)e^{\left(\ln 0.5\right)/5740 \ t} \]

and take the log of both sides to obtain

\[ \ln 0.44 = \left(\ln 0.5\right)/5740 \ t \]

Now solve for \( t \)

\[ t = \left(5740\right)(\ln 0.44) / (\ln 0.5) \approx 6,799 \text{ years} \]
The Definite Integral

Recall that integration has to do with calculating areas.

Now we can be more precise and make a formal definition.

1) Let $y = f(x)$ on a closed interval $[a,b]$
2) Partition the interval into $n$ equal subintervals each of length $\Delta x = (b - a)/n$ and pick any point $w_i$ in the $i^{th}$ subinterval
3) Form the Riemann Sum $\sum f(w_i)\Delta x$ for $i = 1,2,\ldots,n$
4) Now take the limit as $\Delta x \to 0$; that NUMBER, if it exists, is the definite integral

$$\int_a^b f(x) \, dx$$

read “the integral from $a$ to $b$ of $f$ of $x$ dee $x$”
The Definite Integral

Recall that integration has to do with calculating areas.

Now we can be more precise and make a formal definition.

1) Let \( y = f(x) \) on a closed interval \([a,b]\)

2) **Partition** the interval into \( n \) equal subintervals each of length \( \Delta x = \frac{b - a}{n} \) and pick any point \( w_i \) in the \( i^{th} \) subinterval

3) Form the **Riemann Sum** \[ \sum f(w_i) \Delta x \] for \( i = 1,2,...,n \)

4) Now take the limit as \( \Delta x \to 0 \); that NUMBER, if it exists, is the definite integral

\[ \int_{a}^{b} f(x) \, dx \]

read "the integral from \( a \) to \( b \) of \( f \) of \( x \) dee \( x \)"
Some Remarks

The definite integral is a **number**. Finding that number is called **evaluating the integral**. The integral symbol $\int$ is an elongated S which stands for *sum*. The numbers $a$ and $b$ are called the **limits of integration** and $f$ is called the **integrand**. The symbol $dx$ indicates that $x$ is the variable; it varies over the interval $[a,b]$; it is called a *dummy variable* because any letter can be used in its place. Thus,

$$\int_a^b f(x) \, dx, \quad \int_a^b f(t) \, dt \quad \text{and} \quad \int_a^b f(u) \, du$$

are all the same number. The integral defined above is sometimes called the **Riemann Integral**.
As an example, let us compute $\int_0^1 x \, dx$. The integrand is $f(x) = x$ on the closed interval $[0, 1]$.

1) Partition the interval $[0, 1]$ into $n$ equal subintervals $[0, 1/n], [1/n, 2/n], \ldots, [(n-1)/n, 1]$. Thus $\Delta x = (1-0)/n = 1/n$

2) Pick any point $w_i$ in the $i^{th}$ subinterval; let us take it to be the right endpoint: $w_1 = 1/n, w_2 = 2/n, \ldots$

3) Form the Riemann sum

$$\sum f(w_i)\Delta x = f(w_1)(1/n) + f(w_2)(1/n) + \ldots + f(w_n)(1/n)$$

$$= (1/n)(1/n) + (2/n)(1/n) + \ldots + (n/n)(1/n)$$

$$= (1/n^2)(1 + 2 + 3 + \ldots + n)$$

$$= (1/n^2)(n(n + 1)/2) = (n^2 + n)/2n^2$$

$$= (1/2) + (1/2n)$$

4) Take the limit as $\Delta x (= 1/n) \to 0$; the limit is $1/2$ so $\int_0^1 x \, dx = 1/2$ which you will note is the area of the triangle!!
Interpretations of the Definite Integral

First and foremost, the definite integral gives the *area under the curve*.

But each Riemann sum is simply a sum of areas of rectangles and the area of a rectangle, \( A = (\text{height})(\text{base}) \), has many physical interpretations; e.g.,

1) If height is the *speed* of a moving object and the base is time, then the area = speed \( \times \) time = *distance traveled*

2) If the height is the *force* used to move an object and the base is distance, then area = force \( \times \) distance = *work done*

3) If the height is the *rate of consumption of a commodity* and the base is time, then the area = *rate of consumption* \( \times \) time = *total consumption*
Let $f$, $v$, $F$ and $E$ be non-negative continuous functions defined on the closed interval $[a,b]$. Then
\[ \int_a^b f(x) \, dx = \text{area under the curve}. \]

If $v$ is the speed at which an object moves, then
\[ \int_a^b v(t) \, dt = \text{distance traveled}. \]

If $F$ is the force applied to move an object, then
\[ \int_a^b F(x) \, dx = \text{work done}. \]

If $E$ is the consumption rate of a commodity, then
\[ \int_a^b E(t) \, dt = \text{total consumption}. \]
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$$\int_a^b f(x) \, dx = \text{area under the curve.}$$

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If $E$ is the consumption rate of a commodity, then

$$\int_a^b E(t) \, dt = \text{total consumption.}$$
All that and much, much more. BUT we still don’t have a convenient method of evaluating the integral.

Well, that is about to change!!

and here is the basic idea:

\[
y = f(x)\]

For each x in \([a,b]\) let \(G(x) = \) the area under the curve from a to x; ie, let \(G(x) = \int_{a}^{x} f(u) \, du.\)

This is the key that will unlock the magic door.

Lets look at a couple of examples.
Example 1. Let \( y = f(x) = x \) on \([0, 2]\). Find an explicit expression for \( G(x) \).

Now pick any point \( x \) in \([0, 2]\).

Let \( G(x) = \int_0^x u \, du = \) the area under the curve from 0 to \( x \).

This is the area of the triangle with base \( x \) and height \( x \), so \( G(x) = \frac{1}{2}x^2 \).

So, \( f(x) = x \) and \( G(x) = \frac{1}{2}x^2 \). Does that ring a bell?

THAT’S RIGHT!! \( G \) is an antiderivative of \( f \) on \([0, 2]\).

Example 2. Let \( y = f(x) = x - 1 \) on \([2,4]\) and find an explicit expression for \( G(x) \).

Pick any point \( x \) in \([2,4]\)

Let \( G(x) = \int_{2}^{x} (u - 1) \, du = \) the area under the curve from 2 to \( x \).

This is the area of the trapezoid with width \( x - 2 \) and unequal parallel heights of 1 and \( x - 1 \).

The area of a trapezoid is the average of the unequal parallel sides times the distance between them; in this case,

\[
G(x) = \frac{1}{2}(1 + x - 1)(x - 2)
= \frac{1}{2}(x)(x - 2)
= \frac{1}{2}x^2 - x
\]

Once again \( G \) is an antiderivative of \( f \).
THAT’S IT!!! The whole “nine yards,” the “big cahuna,” the “ganze megillah.”

Given a (continuous) function f on a closed interval \([a,b]\), we know that \(G(x) = \int_a^x f(u) \, du\) is an antiderivative of f on \([a,b]\). Now suppose \(F\) is any antiderivative of f on \([a,b]\). By the Equal Derivatives Theorem, we know that \(F\) and \(G\) differ only by a constant; i.e.,

\[
(1) \quad G(x) = F(x) + C \quad \text{for all } x \text{ in } [a,b]
\]

Now \(G(a) = \int_a^a f(u) \, du = 0\), and it follows that

\[
0 = G(a) = F(a) + C \quad \text{so } C = -F(a) \quad \text{and equation (1) becomes}
\]

\[
(2) \quad G(x) = F(x) - F(a) \quad \text{for all } x \text{ in } [a,b]
\]
Recall equation (2) from the previous slide:

\[
(2) \quad G(x) = F(x) - F(a) \quad \text{for all } x \text{ in } [a,b]
\]

From this we obtain

\[
(3) \quad G(b) = F(b) - F(a)
\]

But \( G(b) = \int_a^b f(u) \, du = \int_a^b f(x) \, dx \) and, therefore, equation (3) becomes

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]
We have just “proved”

The Fundamental Theorem of Calculus

Let $f$ be a continuous function on $[a,b]$. Then

(1) $G(x) = \int_a^x f(u) \, du$ is an antiderivative of $f$ on $[a,b]$ and

(2) If $F$ is any antiderivative of $f$ on $[a,b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

That is the **JOY** of calculus
We have just “proved”

The Fundamental Theorem of Calculus

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That is the {JOY} of calculus
Example: Find the area under the curve $y = x^2$ from $a = 0$ to $b = 2$.

Solution: The curve, as we know, is this parabola and this is the area we want to find.

We know that the area is given by $\int_{0}^{2} x^2 \, dx$ and The Fundamental Theorem tells us how to evaluate that integral; find any antiderivative of $x^2$, say $x^3/3$, and then

$$\text{Area} = \int_{0}^{2} x^2 \, dx = 2^3/3 - 0^3/3 = 8/3 \text{ square units.}$$