CIRCLE 6 OF THE FOLLOWING 12 NUMBERS SO THAT THEIR SUM IS 21.

1  1  1
3  3  3
5  5  5
9  9  9
Any questions about the material so far?

Any questions about the Exercises?
1. Given an old fashioned clock, a) through what angle does the hour hand travel in 3 hours? In 30 minutes? b) through what angle does the minute hand travel in 1/3 of an hour? c) what is the angle between the hands at 5:30?

2. Find two complementary angles with one 40° less than the other.

3. Referring to the previous slide about parallel lines, if angle 1 measures 30° what is the measure the other seven angles?

4. Refer again to the parallel Lines. Why does 1 = 7 and why is 4 supplementary to 5?
Triangles

Definition: If any three points A, B, and C do not lie on the same line, then the line segments AB, AC, and BC form a triangle.

Points A, B, and C are called vertices of the triangle and the angles at A, B, and C are called (interior) angles of the triangle.
Theorem: The sum of the angles in any triangle is 180°

Proof: Draw a triangle ABC with angles a, b and c.

1. Draw the line through B parallel to AC and label the angles a’ and c’.

2. Then angle a’ = angle a and angle c’ = angle c. Why?

3. Now angle a’ + angle b + angle c’ = 180°. Why?

4. So angle a + angle b + angle c = 180°.
Types of Triangles

An **acute triangle** is one in which all three angles are acute (less than 90°)

An **obtuse triangle** is one in which one of the angles is obtuse (greater than 90° but less than 180°)

A **right triangle** is one in which one of the angles is a right angle
An **isosceles triangle** is a triangle with two sides of equal length

\[ \triangle \]

An **equilateral triangle** is a triangle with all three sides of equal length

\[ \triangle \]
The term *congruence* is used a lot in Geometry. We say that two geometric figures are *congruent* if one can be made to coincide exactly with the other using only rigid motion; i.e., sliding, rotating or flipping.

In other words, if it were possible (which it isn’t) you could pick up one figure and place it on top of the other and their corresponding parts would line up exactly.

For example, two line segments of the same length are congruent.

Another example is any two circles with the same radius are congruent.
Consider the triangles below

Slide and Rotate

Slide, Rotate and Flip

All three triangles are congruent.
Here are some observations about congruent triangles.

1. Every triangle is congruent to itself. (Reflexive Property)

2. If A and B are both congruent to C, then A is congruent to B (Identity Property)

3. If A is congruent to B and B is congruent to C, then A is congruent to C. (Transitive Property)

4. If A is congruent to B, then their corresponding sides and corresponding angles are congruent.
5. If three sides of triangle A are congruent to the corresponding sides of triangle B, then A and B are congruent. This is known as SSS (side-side-side.)

6. If two sides and the included angle of triangle A are congruent to the corresponding parts of triangle B, then A and B are congruent. This is known as SAS (side-angle-side.)

7. If two angles and the included side of triangle A are congruent to the corresponding parts of triangle B, then A and B are congruent. This is known as ASA (angle-side-angle.)

8. If triangle A has one side and any two angles congruent to the corresponding side and two angles of triangle B, then A and B are congruent. This is known as SAA (side-angle-angle.)

Proof?
Question: What if the three angles of one triangle ABC are congruent to the corresponding angles of another triangle DEF? Are the triangles congruent?

Answer: NO!! Actually there are infinitely many triangles all with the same three angles.
But they do share an important property; namely, that all three sides of one triangle are in the same proportion to the corresponding sides of the other. In this case we say that the triangles are similar. Here is what I mean.

![Diagram of two similar triangles](image)

The corresponding angles in triangles ABC and DEF are equal, so they are similar. Therefore their sides are proportional; i.e., \( \frac{c}{f} = \frac{b}{e} = \frac{a}{d} \) or \( \frac{c}{f} = \frac{b}{e} = \frac{a}{d} \)

(Note the difference with congruence; in congruent triangles, sides must be in proportion 1 to 1.)
Proposition 2

If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle.

Let $DE$ be drawn parallel to $BC$, one of the sides of the triangle $ABC$.

I say that $BD$ is to $AD$ as $CE$ is to $AE$.

Join $BE$ and $CD$.

Therefore the triangle $BDE$ equals the triangle $CDE$, for they are on the same base $DE$ and in the same parallels $DE$ and $BC$.

And $ADE$ is another triangle.

But equals have the same ratio to the same, therefore the triangle $BDE$ is to the triangle $ADE$ as the triangle $CDE$ is to the triangle $ADE$.

But the triangle $BDE$ is to $ADE$ as $BD$ is to $AD$, for, being under the same height, the perpendicular drawn from $E$ to $AB$, they are to one another as their bases.

For the same reason, the triangle $CDE$ is to $ADE$ as $CE$ is to $AE$.

Therefore $BD$ is to $AD$ also as $CE$ is to $AE$. 
Proposition 4

In equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.

Let $ABC$ and $DCE$ be equiangular triangles having the angle $ABC$ equal to the angle $DCE$, the angle $BAC$ equal to the angle $CDE$, and the angle $ACB$ equal to the angle $CED$.

I say that in the triangles $ABC$ and $DCE$ the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.

Let $BC$ be placed in a straight line with $CE$.

Then, since the sum of the angles $ABC$ and $ACB$ is less than two right angles, and the angle $ACB$ equals the angle $DEC$, therefore the sum of the angles $ABC$ and $DEC$ is less than two right angles. Therefore $BA$ and $ED$, when produced, will meet. Let them be produced and meet at $F$.

Now, since the angle $DCE$ equals the angle $ABC$, $DC$ is parallel to $FB$. Again, since the angle $ACB$ equals the angle $DEC$, $AC$ is parallel to $FE$.

Therefore $FACD$ is a parallelogram, therefore $FA$ equals $DC$, and $AC$ equals $FD$.

And, since $AC$ is parallel to a side $FE$ of the triangle $FBE$, therefore $BA$ is to $AF$ as $BC$ is to $CE$.

But $FD$ equals $AC$, therefore $BC$ is to $CE$ as $AC$ is to $DE$, and alternately $BC$ is to $CA$ as $CE$ is to $ED$.

Since then it was proved that $AB$ is to $BC$ as $DC$ is to $CE$, and $BC$ is to $CA$ as $CE$ is to $ED$, therefore, ex aequali, $BA$ is to $AC$ as $CD$ is to $DE$.

Therefore, in equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.

Q.E.D.
Theorem: In a right triangle, if a line segment is drawn from the vertex of the right angle perpendicular to the hypotenuse, then all three triangles formed are similar.

Proof: Let’s label the pertinent angles.
Then 1 is congruent to 5, 2 is congruent to 6 and 3 is congruent 4. Why?
Thus, the two smaller triangles are similar and by a like argument each of the smaller triangles is similar to the larger one.
Now I can show you a really neat proof of the Pythagorean Theorem: In a right triangle with legs $a$ and $b$ and hypotenuse $c$, then $a^2 + b^2 = c^2$

![Diagram of a right triangle with legs $a$ and $b$ and hypotenuse $c$, and a segment from the vertex of the right angle perpendicular to the hypotenuse labeled $p$ and $q$.]

Proof. Draw a right triangle with legs $a$ and $b$ and hypotenuse $c$. Draw a segment from the vertex of the right angle perpendicular to the hypotenuse and label the parts $p$ and $q$. The triangles are similar so their sides are proportional; i.e., $c/a = a/p$ and $c/b = b/q$. From these equations, it follows that $a^2 = cp$ and $b^2 = cq$ so

$$a^2 + b^2 = cp + cq = c(p + q) = c^2 \quad \text{QED}$$
Problem: Given two right triangles, the hypotenuse of each is congruent and one leg of each is congruent. Are the triangles necessarily congruent?
Theorem: In an isosceles triangle ABC, the angles opposite the congruent sides AB and BC are also congruent.

Proof.
1. Draw an isosceles triangle ABC and draw a segment BD from B to side AC that bisects angle B. Label the two congruent angles 1 and 2 as shown below.

2. We now have two triangles ABD and DBC. We are given that AB is congruent to BC. By construction, angle 1 is congruent to angle 2 and surely BD is congruent to itself.

3. Therefore, by SAS, triangle ABD is congruent to triangle DBC so angle A is congruent to angle C.
Exercises
1. Show that if two angles of a triangle are congruent, then the sides opposite those angles are also congruent.
2. Show that in an equilateral triangle, all three angles are congruent. What is their measure?
3. Draw an isosceles triangle ABC with AB congruent to BC. Prove that the line that bisects angle B is also the perpendicular bisector of side AC.
4. If two line segments AD and BE meet at point C and bisect each other, prove that the line segments AB and DE are congruent.
5. In triangle ABC, a line segment from a point D on side AB to a point E on side BC is parallel to side AC. Show that the two triangles formed are similar so that their sides are proportional.
6. Prove that every point on the perpendicular bisector of segment AB is equidistant from A and B.